

EQUIVARIANT K -THEORY OF FLAG VARIETIES REVISITED AND RELATED RESULTS

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ABSTRACT. In this article we obtain many results on the multiplicative structure constants of T -equivariant Grothendieck ring $K_T(G/B)$ of the flag variety G/B . We do this by lifting the classes of structure sheaves of Schubert varieties in $K_T(G/B)$ to $R(T) \otimes R(T)$, where $R(T)$ denotes the representation ring of the torus T . We further apply our results to describe the multiplicative structure constants of $K(X)_{\mathbb{Q}}$ where X denotes the wonderful compactification of the adjoint group of G , in terms of the structure constants of Schubert varieties in the Grothendieck ring of G/B .

1. INTRODUCTION

Let G be a semi-simple simply connected algebraic group over an algebraically closed field k . Let B be a Borel subgroup and $T \subset B$ be a maximal torus.

In this article we construct explicit lifts of the classes of the structure sheaves of Schubert basis in $K_T(G/B)$ in the ring $R(T) \otimes_{\mathbb{Z}} R(T)$. For this, we apply techniques similar to those developed in the paper of Marlin (see [14]) by exploiting the properties of Demazure operators ([7]).

Using these lifts we also give new methods to describe the multiplicative structure of $K_T(G/B)$. More precisely, in §2 and §3, we give closed formulas for the multiplicative structure constants and also recover some known results on these constants in this setting.

This was inspired by the results of Hiller (see Chapter IV of [13]) who constructs basis for $Sym(X^*(T))$ as $Sym(X^*(T))^W$ -module by lifting the fundamental classes of Schubert basis in the cohomology ring $H^*(G/B)$. He further uses this basis to develop an algebraic approach to Schubert calculus in $H^*(G/B)$.

In §4, we too show that the above constructed lifts of the Schubert basis in $K(G/B)$ to $R(T)$, form a basis of $R(T)_{\mathfrak{p}}$ over $R(G)_{\mathfrak{p}}$ where \mathfrak{p} denotes the kernel of the augmentation map $R(G) \rightarrow \mathbb{Z}$. These are a new set of basis for $R(T)_{\mathfrak{p}}$ as $R(G)_{\mathfrak{p}}$ -module different from the basis obtained by localization from that defined by Steinberg in [19].

In the final section we apply this result to the description of $K(X)$ the Grothendieck ring of wonderful compactification X of the semi-simple adjoint group $G_{ad} = G/Z(G)$. In fact this ring has been studied earlier in [21]. Indeed, in the main result of [21], the images in $K(G/B)$ under c_K of the Steinberg basis of $R(T)$ as an $R(G)$ -module, are used to describe the multiplicative structure of $K(X)$. Thus the multiplicative structure constants of $K(X)$ as a $K(G/B)$ -algebra involved the images of the structure constants of the Steinberg basis which do not have known geometric or representation theoretic interpretations.

In the main result of this paper we show that $K(X)_{\mathbb{Q}}$ is a free $K(G/B)$ -module generated by classes of the structure sheaves of Schubert basis in all $K(G/P)_{\mathbb{Q}}$, where $P \supseteq B$ is a parabolic subgroup. In particular, we express the multiplicative structure constants of $K(X)_{\mathbb{Q}}$ as $K(G/B)$ -algebra in terms of the structure constants of the classes of structure sheaves of Schubert varieties in the Grothendieck ring of flag varieties.

Thus although we seem to lose to some extent by going to rational coefficients, we do finally gain in terms of getting better interpretations of the basis and the multiplicative structure of the $K(X)_{\mathbb{Q}}$, by relating it to the Schubert calculus in the Grothendieck ring of flag varieties.

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1.1. Notations and Conventions. Let $\Lambda = X^*(T)$ denote the weight lattice. Let Φ denote the root system and Δ the set of simple roots relative to B . Let $W = N(T)/T$ be the Weyl group of the root system Φ . Let $B^- = w_0 B w_0$ be the opposite Borel subgroup to B where w_0 is the unique maximal element of the Bruhat order on W . Let $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. Let ω_α denote the fundamental weight corresponding to the simple root $\alpha \in \Delta$.

For $w \in W$, let X_w denote the Schubert variety which is the closure of the Schubert cell BwB/B in G/B and let X^w denote opposite Schubert variety which is the closure of the opposite Schubert cell B^-wB/B in G/B . Thus we have: $X^w = w_0 X_{w_0 w}$.

Let $v, w \in W$. Recall that $X^v \cap X_w$ is nonempty if and only if $v \preceq w$; then $X^v \cap X_w$ is a variety called the Richardson variety and is denoted by X_w^v . Moreover, X_w^v has two kinds of boundaries, namely $(\partial X_w)^v := (\partial X_w) \cap X^v$ and $(\partial X^v)_w := (\partial X^v) \cap X_w$. Here $\partial X_w = \bigcup_{w' \prec w} X_{w'}$ is the boundary of the Schubert variety X_w and $\partial X^v = \bigcup_{v \prec v'} X^{v'}$ is the boundary of the opposite Schubert variety X^v . Thus we have $(\partial X_w)^v = \bigcup_{w' \prec w} X_{w'}^v$ and $(\partial X^v)_w = \bigcup_{v \prec v'} X_w^{v'}$ (see Prop.1.3.2 and §4.2 of [4]).

For X any smooth G -variety, let $K_G(X)$ denote the Grothendieck ring of G -equivariant coherent sheaves (or equivalently, vector bundles) on X . We have the canonical forgetful homomorphism $K_G(X) \rightarrow K_T(X)$. In particular, $R(G) := K_G(pt)$ is the Grothendieck ring of k -representations of G . Since G is simply connected, we can identify $R(G) = \mathbb{Z}[\Lambda]^W$ via restriction to T . Furthermore, the structure morphism $X \rightarrow k$ induces a canonical $R(G)$ -module structure on $K_G(X)$. Also, $K(X)$ denotes the Grothendieck ring of coherent sheaves on X and we have canonical forgetful homomorphism $K_G(X) \rightarrow K(X)$.

For $w \in W$, let $[\mathcal{O}_{X_w}]_T$ (resp. $[\mathcal{O}_{X^w}]_T$) denote the class of the structure sheaf of the Schubert variety (resp. opposite Schubert variety) in $K_T(G/B)$. Further, note that we have the identification $[\mathcal{O}_{X^w}]_T = w_0 \cdot [\mathcal{O}_{X_{w_0 w}}]_T$ in $K_T(G/B)$. Recall from [16] that the Schubert classes $\{[\mathcal{O}_{X^w}]_T\}_{w \in W}$ form a basis of $K_T(G/B)$ as an $R(T)$ -module.

For $\lambda \in \Lambda$, let $\mathcal{L}^T(\lambda)$ denote the T -linearized line bundle on G/B associated to λ . Let $c_K^T : \mathbb{Z}[\Lambda] = R(T) \rightarrow K_T(G/B)$ denote the characteristic homomorphism which sends $e^\lambda \in R(T)$ to $[\mathcal{L}^T(\lambda)] \in K_T(G/B)$.

Let $*$ denote the canonical involution in $K_T(G/B)$ defined by duality of a T -vector bundle. This is compatible with the involution in $R(T)$ defined by $e^\lambda \mapsto e^{-\lambda}$.

Recall that Demazure has defined the operators L_w , $w \in W$, on $\mathbb{Z}[\Lambda]$ satisfying the following properties:

$$(1.1) \quad \begin{aligned} L_w L_{w'} &= L_{ww'} && \text{if } l(ww') = l(w) + l(w') \\ L_s L_s &= L_s && \text{if } l(s) = 1 ; \text{ i.e } s = s_\alpha \text{ for some } \alpha \in \Delta \\ L_{s_\alpha}(e^\lambda) &= \frac{e^\lambda - e^{s_\alpha(\lambda)}}{1 - e^\alpha} && \text{for } \alpha \in \Delta \end{aligned}$$

(see Theorem 2, pp. 86-87 of [7]). Also, we have:

$$(1.2) \quad L_s \cdot L_w = \begin{cases} L_{sw} & \text{if } l(sw) = l(w) + 1 \\ L_w & \text{if } l(sw) = l(w) - 1 \end{cases}$$

and for any $w' \in W$, there exists a unique $\overline{w} \in W$ such that:

$$(1.3) \quad L_{w'} L_{w^{-1}w_0} = L_{\overline{w}}$$

(see §5.6 of [7]).

Let \mathcal{F} be a T -equivariant coherent sheaf on G/B . We define the equivariant Euler-Poincaré characteristic of $[\mathcal{F}] \in K_T(X_w)$ as:

$$(1.4) \quad \chi^T(X_w, [\mathcal{F}]) = \sum_k (-1)^k \text{Char}(H^k(X_w, \mathcal{F})) \in R(T)$$

where $\text{Char}(H^k(X_w, \mathcal{F})) \in R(T)$ is the character of the finite dimensional T -module $H^k(X_w, \mathcal{F}) = H^k(G/B, \mathcal{O}_{X_w} \otimes \mathcal{F})$. Then $\chi^T : K_T(X_w) \rightarrow R(T)$ is an $R(T)$ -linear map. Further,

$$(1.5) \quad \chi^T(X_w, c_K^T(e^\lambda)) = e^\rho \cdot L_w(e^{\lambda-\rho}).$$

Moreover, if $\epsilon : R(T) \rightarrow \mathbb{Z}$ denotes the canonical augmentation, then

$$(1.6) \quad \chi(X_w, \mathcal{L}(\lambda)) = \epsilon L_w(e^{\lambda-\rho})$$

where $\chi(.,.)$ denotes the ordinary Euler-Poincaré characteristic (see Theorem 2(b) and Cor.1 pp. 86-87 of [7]).

In [16], Kostant and Kumar define an $R(T)$ -module basis $(\tau^w)_{w \in W}$ for $K_T(G/B)$ which satisfies:

$$(1.7) \quad \chi^T(X_{v^{-1}}, * \tau^w) = \delta_{v,w}$$

(see p. 591, Prop.3.39 of [16]).

In [10], Graham and Kumar define the elements $\xi^w = [\mathcal{O}_{X^w}(-\partial X^w)]_T$ which form an $R(T)$ -basis $\{\xi^w\}_{w \in W}$ for $K_T(G/B)$ dual to the Schubert basis $\{[\mathcal{O}_{X_w}]_T\}_{w \in W}$ under the pairing:

$$(1.8) \quad \langle u, v \rangle := \chi^T(G/B, u \cdot v) \text{ for all } u, v \in K_T(G/B).$$

Further, it is shown in Prop. 2.2 of [10] that:

$$(1.9) \quad * \tau^w = \xi^{w^{-1}}$$

where τ^w is the Kostant-Kumar basis.

We further have the following relation between the Graham-Kumar basis and the opposite Schubert basis of $K_T(G/B)$ (see [10]):

$$(1.10) \quad [\mathcal{O}_{X^w}]_T = \sum_{w \preceq w'} \xi^{w'}.$$

For $I \subseteq \Delta$, let W_I be the subgroup of W generated by $\{s_\alpha : \alpha \in I\}$. Further, let W^I denote the minimal length coset representatives of W/W_I . Let $P = P_I \supset B$ denote the corresponding standard parabolic subgroup. In particular, for $w \in W^I$, we have the Schubert variety X_w^P (resp. the opposite Schubert variety X_P^w) which is the closure of the Bruhat cell BwP/P (resp. opposite Bruhat cell B^-wP/P) in the partial flag variety G/P .

It is well known that $\{[\mathcal{O}_{X_w^P}]_T\}_{w \in W^I}$ is an $R(T)$ -basis of $K_T(G/P)$, and so is $\{[\mathcal{O}_{X_P^w}]_T\}_{w \in W^I}$. Further, in §2 of [10], Graham and Kumar define the elements:

$$(1.11) \quad \xi_P^v = [\mathcal{O}_{X_P^v}(-\partial X_P^v)]_T$$

which form an $R(T)$ -basis $\{\xi_P^v\}_{v \in W^I}$ for $K_T(G/P)$ dual to the Schubert basis $\{[\mathcal{O}_{X_w^P}]_T\}_{w \in W^I}$ under the pairing:

$$(1.12) \quad \langle [\mathcal{O}_{X_w^P}]_T, \xi_P^v \rangle = \chi^T(G/P, \mathcal{O}_{X_w^P \cap X_P^v}(-X_w^P \cap \partial X_P^v)).$$

Note that (1.12) is a generalization of (1.8) to G/P .

Let $K(G/B)$ denote the Grothendieck ring of coherent sheaves on G/B . Further,

$$c_K : \mathbb{Z}[\Lambda] = R(T) = K_G(G/B) \rightarrow K(G/B)$$

denote the characteristic homomorphism. Recall that in [7], Demazure has established the existence of a basis $(a_w)_{w \in W}$ for the \mathbb{Z} -module $K(G/B)$ such that:

$$(1.13) \quad c_K(e^\lambda) = \sum_{w \in W} \chi(X_w, \mathcal{L}(\lambda)) a_w$$

where $\chi(.,.)$ denotes the Euler-Poincaré characteristic.

If $\epsilon : \mathbb{Z}[\Lambda] \rightarrow \mathbb{Z}$ denotes the canonical augmentation, then

$$(1.14) \quad \chi(X_w, \mathcal{L}(\lambda)) = \epsilon L_w(e^{\lambda - \rho})$$

where L_w is as in (1.1).

Recall that we have the following relation between the Demazure basis and Schubert basis:

$$(1.15) \quad [\mathcal{O}_{X^w}] = \sum_{w \preceq w'} a_{w'}.$$

Let

$$(1.16) \quad f : K_T(G/B) \rightarrow K(G/B)$$

denote the forgetful homomorphism. Then we have $f([\mathcal{O}_{X^w}]_T) = [\mathcal{O}_{X^w}]$ and $f(\xi^w) = a_w$ (see Prop. 3.39 of [16]).

2. EQUIVARIANT K -THEORY OF FLAG VARIETIES

2.1. Lifting of Schubert basis to $R(T) \otimes R(T)$. In this section we construct explicit lifts of classes of structure sheaves of Schubert varieties in $K_T(G/B)$ to the ring $R(T) \otimes R(T)$. These shall be used in the later sections to describe the multiplicative structure of $K_T(G/B)$.

We mention here that tensor products are considered over \mathbb{Z} unless specified.

Lemma 2.1.. *The canonical homomorphism*

$$\Psi : R(T) \otimes R(T) \rightarrow K_T(G/B)$$

that sends an element $\sum_{i=1}^n a_i \otimes b_i$ in $R(T) \otimes R(T)$ to the element $\sum_{i=1}^n a_i \cdot c_K^T(b_i)$ in $K_T(G/B)$ is surjective with kernel the ideal

$$\mathcal{I} = \langle c \otimes 1 - 1 \otimes c : c \in R(T)^W \rangle$$

in $R(T) \otimes R(T)$.

Proof: We recall from Prop. 4.1 of [15] that the map:

$$(2.17) \quad R(T) \otimes_{R(G)} K_G(G/B) = R(T) \otimes_{R(T)^W} R(T) \rightarrow K_T(G/B),$$

defined via $a \otimes b \mapsto a \cdot c_K^T(b)$ is an isomorphism. Moreover, by definition of $R(T) \otimes_{R(T)^W} R(T)$, there is canonical surjective homomorphism $\psi : R(T) \otimes R(T) \rightarrow R(T) \otimes_{R(T)^W} R(T)$ with kernel precisely \mathcal{I} . Now, Ψ is the homomorphism obtained by composing ψ with the isomorphism given by (2.17). Thus it follows that Ψ is surjective with kernel \mathcal{I} . (Also see Theorem 1.2 of [11]). \square

Definition 2.2.. *By defining $\mathbb{L}_w(a \otimes b) := a \otimes L_w(b)$ and extending it by linearity, we can define the Demazure operator \mathbb{L}_w on $R(T) \otimes R(T)$ as an $R(T) \otimes 1$ -linear operator.*

We now prove some preliminary lemmas which shall be applied in the main proposition.

Lemma 2.3.. *If $w \preceq w'$ then $L_{w'} L_{w^{-1}w_0} = L_{w_0}$.*

Proof: We note that when $l(w) = 0$ then $w = 1$. Thus by (1.2) we have:

$$(2.18) \quad L_{w'} L_{w^{-1}w_0} = L_{w'} L_{w_0} = L_{w_0}.$$

Also, when $l(w') - l(w) = 0$ then $w = w'$. Again by (1.2) we have:

$$(2.19) \quad L_{w'} L_{w^{-1}w_0} = L_w L_{w_r} = L_{w_0}.$$

We shall now prove the claim by induction on $l(w)$ and $l(w') - l(w)$.

Let $s'_1 \cdots s'_k$ be a reduced expression of w' and let $w = s'_{i_1} \cdots s'_{i_m}$. Now, we can write:

$$(2.20) \quad L_{w'} L_{w^{-1}w_0} = L_v L_{s'_k} L_{w^{-1}w_0}$$

where $v = s'_1 \cdots s'_{k-1}$.

Case (i) If $l(s'_k w_r) = l(w_r) - 1$ then by (1.2) we have:

$$(2.21) \quad L_v L_{s'_k} L_{w^{-1}w_0} = L_v L_{w_r} = L_v L_{w^{-1}w_0}.$$

Moreover, we note that $l(s'_k w_r) = l(w_r) - 1$ is equivalent to $l(ws'_k) = l(w) + 1$. This further implies that $i_m \leq k - 1$. Hence it follows that $w \preceq v$. Now, since $l(v) - l(w) \leq l(w') - l(w)$, the claim follows by induction on $l(w') - l(w)$.

Case (ii) If $l(s'_k w_r) = l(w_r) + 1$ then again by (1.2) we have:

$$(2.22) \quad L_v L_{s'_k} L_{w^{-1}w_0} = L_v L_{s'_k w_r} = L_v L_{(ws'_k)^{-1}w_0}.$$

Note that $l(s'_k w_r) = l(w_r) + 1$ implies $l(ws'_k) = l(w) - 1$. Since $w \preceq w'$, this further implies that $ws'_k \preceq w'$ (see Proposition on p.119 of [12]). Moreover, since $l(ws'_k) \leq l(w)$, the claim now follows by induction on $l(w)$. Hence the proof. \square

Lemma 2.4.. *Let \bar{w} be as in (1.3). Then*

$$(2.23) \quad \bar{w} = w_0 \Leftrightarrow w \preceq w'.$$

Proof: Let $l(w) = r$ and $w_r := w^{-1}w_0$. Let $w' = s'_1 \cdots s'_k$ be a reduced expression for w' . Hence

$$(2.24) \quad L_{w'} \cdot L_{w^{-1}w_0} = L_{w'} \cdot L_{w_r} = L_{s'_1} \cdots L_{s'_k} \cdot L_{w_r}.$$

Now, by (1.2) we see that:

$$(2.25) \quad L_{s'_1} \cdots L_{s'_k} \cdot L_{w_r} = L_{s'_{i_1} s'_{i_2} \cdots s'_{i_m} w_r}$$

for some subsequence $(s'_{i_1}, \dots, s'_{i_m})$ of (s'_1, \dots, s'_k) . Now, (1.3), (2.24) and (2.25) imply that

$$(2.26) \quad \bar{w} = s'_{i_1} s'_{i_2} \cdots s'_{i_m} w_r.$$

Since $w_0 = w \cdot w_r$, it follows that $\bar{w} = w_0$ will imply:

$$(2.27) \quad w = s'_{i_1} s'_{i_2} \cdots s'_{i_m}.$$

Hence we get: $w \preceq w'$. Converse follows by Lemma 2.3. Hence the proof. \square

Proposition 2.5.. *In $R(T) \otimes R(T)$ there exists an element u_0 such that*

$$(2.28) \quad \Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho)) = [\mathcal{O}_{X^w}]_T.$$

Indeed, we may take $u_0 = v_0(1 \otimes e^{-\rho})$ where v_0 is such that $\Psi(v_0) = [\mathcal{O}_{X^{w_0}}]_T$.

Proof: By (1.7) and (1.9) we have the following identity in $K_T(G/B)$:

$$(2.29) \quad c_K^T(e^\lambda) = \sum_{w \in W} \chi^T(X_w, \mathcal{L}(\lambda)) \xi^w.$$

Moreover, combining (1.5) and (2.29) it also follows that:

$$(2.30) \quad c_K^T(e^\lambda) = \sum_{w \in W} e^\rho \cdot L_w(e^{\lambda-\rho}) \xi^w.$$

By (1.10), it follows in particular that $\xi^{w_0} = [\mathcal{O}_{X^{w_0}}]_T = w_0 \cdot [\mathcal{O}_{X_1}]_T$.

Now, since Ψ is surjective by Lemma 2.1, there exists an element v_0 such that $\Psi(v_0) = \xi^{w_0}$. More precisely, if

$$(2.31) \quad v_0 = \sum_{i=1}^n a_i \otimes b_i,$$

then using (2.30) we have:

$$(2.32) \quad \Psi(v_0) = \sum_{i=1}^n a_i \cdot c_K^T(b_i) = \sum_{i=1}^n \cdot \sum_{w \in W} a_i \cdot e^\rho \cdot L_w(b_i \cdot e^{-\rho}) \xi^w.$$

Hence we see that:

$$(2.33) \quad \Psi(v_0) = \sum_{w \in W} \sum_{i=1}^n a_i \cdot e^\rho \cdot L_w(b_i \cdot e^{-\rho}) \xi^w = \xi^{w_0}.$$

Therefore we have:

$$(2.34) \quad \sum_{i=1}^n a_i \cdot e^\rho \cdot L_w(b_i \cdot e^{-\rho}) = \delta_{w, w_0}.$$

Let

$$(2.35) \quad u_0 := v_0 \cdot (1 \otimes e^{-\rho}).$$

Claim: u_0 is the required element in $R(T) \otimes R(T)$ that satisfies (2.28).

Proof of Claim: Note that if v_0 is as in (2.31) then:

$$(2.36) \quad u_0 = \sum_{i=1}^n a_i \otimes e^{-\rho} \cdot b_i.$$

Now, by (2.36) and Def.(2.2) it follows that:

$$(2.37) \quad \mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho) = \sum_{i=1}^n a_i \otimes e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot b_i).$$

Hence by (2.32) and (2.37) we have:

$$(2.38) \quad \Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho)) = \sum_{w' \in W} e^\rho \cdot a_i \cdot L_{w'} L_{w^{-1}w_0}(b_i \cdot e^{-\rho}) \xi^{w'}.$$

Now we see that the claim follows by (1.10), (2.34), (2.38) and Lemma 2.4. \square

Lemma 2.6.. *If $w \in W^I$ and $r \in R(T)$, then we have:*

$$(2.39) \quad e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r) \in R(T)^{W_I}.$$

Proof: We first note that:

$$(2.40) \quad s_j(e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r)) = e^{\rho - \alpha_i} \cdot s_j(L_{w^{-1}w_0}(e^{-\rho} \cdot r)).$$

Thus we see that for $j \in I$, the condition:

$$(2.41) \quad s_j(e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r)) = e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r)$$

is equivalent to:

$$(2.42) \quad s_j(L_{w^{-1}w_0}(e^{-\rho} \cdot r)) = e^{\alpha_i} \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r).$$

Further, note that

$$(2.43) \quad s_j(L_{w^{-1}w_0}(e^{-\rho} \cdot r)) = L_{w^{-1}w_0}(e^{-\rho} \cdot r) - (1 - e^{\alpha_i}) \cdot L_{s_j} L_{w^{-1}w_0}(e^{-\rho} \cdot r).$$

Let $w_1 := w^{-1}w_0$. Then we have

$$(2.44) \quad l(s_j w_1) = l(s_j w^{-1} w_0) = l(w_0) - l(s_j w^{-1}).$$

Now, if $w \in W^I$, then for every $j \in I$ we have $l(ws_j) = l(w) + 1$ which is equivalent to $l(s_j w^{-1}) = l(w^{-1}) + 1$. This implies by (2.44) that:

$$(2.45) \quad \begin{aligned} l(s_j w_1) &= l(w_0) - (l(w^{-1}) + 1) \\ &= l(w_0) - l(w^{-1}) - 1 \\ &= l(w^{-1} w_0) - 1 \\ &= l(w_1) - 1. \end{aligned}$$

This further implies by (1.2) that:

$$(2.46) \quad L_{s_j} L_{w^{-1}w_0} = L_{s_j} L_{w_1} = L_{w_1}.$$

Now by substituting (2.46) in (2.43), we see that when $w \in W^I$, the condition (2.42) and hence (2.41) hold for all $j \in I$. This proves that if $w \in W^I$ then $e^\rho \cdot L_{w^{-1}w_0}(e^{-\rho} \cdot r) \in R(T)^{W_I}$. \square

Proposition 2.7.. *Let u_0 be as in Prop. 2.5. Then the element*

$$\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho)$$

belongs to $R(T) \otimes R(T)^{W_I}$ if $w \in W^I$.

Proof: This proposition follows immediately from (2.37) and Lemma 2.6. \square .

Notation 2.8.. In the following sections we let $u_0 = \sum_{i=1}^n a_i \otimes e^{-\rho} \cdot b_i \in R(T) \otimes R(T)$ be as in Prop.2.5. (See the concluding Remark 5.2 about this choice.)

2.2. Structure constants of Schubert basis in $K_T(G/B)$. In this section we determine a closed formula for the multiplicative structure constants of the basis $\{[\mathcal{O}_{X^w}]_T\}_{w \in W}$ in $K_T(G/B)$ in terms of the above elements a_i, b_i . We remark here that in [10], these structure constants as well as those of the dual basis have been studied in detail with regard to the positivity conjectures viz., Conjecture 3.1 and 3.10 of [10]. The author is currently working to find more direct inter-connections between the results in this section and those in [10].

Lemma 2.9.. *For $x, y, z \in W$, let*

$$(2.47) \quad C_{x,y}^z := \sum_{w \preceq z} (-1)^{l(z)-l(w)} \sum_{1 \leq i,j \leq n} a_i \cdot a_j \cdot e^\rho \cdot L_w(L_{x^{-1}w_0}(b_i \cdot e^{-\rho}) \cdot L_{y^{-1}w_0}(b_j \cdot e^{-\rho}) \cdot e^\rho)$$

where $v_0 = \sum_{i=1}^n a_i \otimes b_i \in R(T) \otimes R(T)$ is such that $\Psi(v_0) = [\mathcal{O}_{X^{w_0}}]_T$. Then in $K_T(G/B)$ we have:

$$(2.48) \quad [\mathcal{O}_{X^x}]_T [\mathcal{O}_{X^y}]_T = \sum_{z \in W} C_{x,y}^z [\mathcal{O}_{X^z}]_T$$

for $x, y \in W$.

Proof: Recall from Lemma 4.2 of [10] that the basis $\{\xi^v\}_{v \in W}$ can be expressed in terms of the Schubert basis $\{[\mathcal{O}_{X^v}]_T\}_{v \in W}$ in $K_T(G/B)$ as follows:

$$(2.49) \quad \xi^v = \sum_{v \preceq w} (-1)^{l(w)-l(v)} [\mathcal{O}_{X^w}]_T.$$

Note that (2.49) is equivalent to (1.10) via Möbius inversion. Now, using Lemma 2.1 and substituting (2.49) in (2.30) we get:

$$(2.50) \quad \Psi(a \otimes b) = \sum_{w \in W} \sum_{v \in W, v \preceq w} (-1)^{l(w)-l(v)} a \cdot e^\rho \cdot L_v(b \cdot e^{-\rho}) [\mathcal{O}_{X^w}]_T$$

for $a \otimes b \in R(T) \otimes R(T)$.

Moreover, by (2.37), $\mathbb{L}_{x^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho) \cdot \mathbb{L}_{y^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho) =$

$$(2.51) \quad \sum_{1 \leq i, j \leq n} a_i \cdot a_j \otimes e^{2\rho} \cdot L_{x^{-1}w_0}(e^{-\rho} \cdot b_i) \cdot L_{y^{-1}w_0}(e^{-\rho} \cdot b_j).$$

Further, by Prop. 2.5 it follows that:

$$(2.52) \quad \Psi(\mathbb{L}_{x^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho) \cdot \mathbb{L}_{y^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho)) = [\mathcal{O}_{X^x}]_T [\mathcal{O}_{X^y}]_T$$

for $x, y \in W$. Then by (2.52) and (2.50) we get (2.48) where $C_{x,y}^z$ is as in (2.47). \square

2.3. A Chevalley formula in $K_T(G/B)$. The following lemma gives “a Chevalley formula” in $K_T(G/B)$, which determines the coefficients when the product $[\mathcal{L}^T(\lambda)]_T [\mathcal{O}_{X^x}]_T$ is expressed in terms of the Schubert basis $\{[\mathcal{O}_{X^v}]_T : v \in W\}$.

Lemma 2.10.. *For $\lambda \in X^*(T)$ and $x, y \in W$ let*

$$(2.53) \quad Q_{x,y}^\lambda := \sum_{w \in W, w \preceq y} (-1)^{l(y)-l(w)} \sum_{i=1}^n e^\rho \cdot a_i \cdot L_w(e^\lambda \cdot L_{x^{-1}w_0}(b_i \cdot e^{-\rho}))$$

where $v_0 = \sum_{i=1}^n a_i \otimes b_i \in R(T) \otimes R(T)$ is such that $\Psi(v_0) = [\mathcal{O}_{X^{w_0}}]_T$. Then in $K_T(G/B)$ we have:

$$(2.54) \quad [\mathcal{L}^T(\lambda)]_T \cdot [\mathcal{O}_{X^x}]_T = \sum_{y \in W} Q_{x,y}^\lambda [\mathcal{O}_{X^y}]_T.$$

Proof: Note that

$$(2.55) \quad \Psi(1 \otimes e^\lambda) = c_K^T(e^\lambda) = [\mathcal{L}^T(\lambda)]_T.$$

By (2.55) and Prop.2.5 it follows that:

$$(2.56) \quad \Psi(\mathbb{L}_{x^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho) \cdot (1 \otimes e^\lambda)) = [\mathcal{O}_{X^x}]_T \cdot [\mathcal{L}^T(\lambda)]_T$$

for $x \in W$ and $\lambda \in X^*(T)$. Thus we see that (2.54) follows immediately from (2.37), (2.50) and (2.56) where $Q_{x,y}^\lambda$ is given by (2.53). \square

2.3.1. Comparison with known Chevalley formulas.

Lemma 2.11.. *Let $x, y \in W$ and $w = w_0x$, $v = w_0y$. We then have the following interpretation of (2.53):*

$$(2.57) \quad Q_{x,y}^\lambda = w_0 \cdot \chi^T(X_w^v, \mathcal{L}^T(w_0(\lambda))(-(\partial X^v)_w))$$

whenever $x \preceq y$ and $Q_{x,y}^\lambda = 0$ otherwise. In particular, when $w_0(\lambda) \in X^*(T)$ is dominant we have:

$$(2.58) \quad Q_{x,y}^\lambda = w_0 \cdot \text{Char } H^0(X_w^v, \mathcal{L}^T(w_0(\lambda))(-(\partial X^v)_w))$$

whenever $x \preceq y$ and $Q_{x,y}^\lambda = 0$ otherwise.

Proof: From (1.8) and (2.54) it follows that:

$$(2.59) \quad \begin{aligned} Q_{x,y}^\lambda &= \langle [\mathcal{L}^T(\lambda)]_T \cdot [\mathcal{O}_{X^x}]_T, \xi^y \rangle \\ &= \chi^T(X_y^x, [\mathcal{L}^T(\lambda)]_T(-\partial X^x)_y) \end{aligned}$$

whenever $x \preceq y$, and $Q_{x,y}^\lambda = 0$ otherwise.

Now, if $w = w_0x$ and $v = w_0y$, we can write (2.59) as:

$$(2.60) \quad \begin{aligned} Q_{x,y}^\lambda &= \chi^T(X_{w_0v}^{w_0w}, \mathcal{L}^T(\lambda)(-(\partial X_{w_0v})^{w_0w})) \\ &= \chi^T(w_0 \cdot X_w^v, \mathcal{L}^T(\lambda)(-w_0 \cdot (\partial X^v)_w)) \\ &= w_0 \cdot \chi^T(X_w^v, \mathcal{L}^T(w_0(\lambda))(-(\partial X^v)_w)). \end{aligned}$$

whenever $v \preceq w$ and $Q_{x,y}^\lambda = 0$ otherwise. (Note that $x \preceq y$ is equivalent to $v \preceq w$.) In particular, when $w_0(\lambda) \in X^*(T)$ is dominant, by Prop.1 on p.9 of [5] it follows that:

$$\chi^T(X_w^v, \mathcal{L}^T(w_0(\lambda))(-(\partial X^v)_w)) = \text{Char } H^0(X_w^v, \mathcal{L}^T(w_0(\lambda))(-(\partial X^v)_w)).$$

Hence the lemma. \square

Remark 2.12.. Let $w = w_0x$ and $v = w_0y$. Then (2.54) can be rewritten as:

$$(2.61) \quad [\mathcal{L}^T(w_0\lambda)]_T \cdot [\mathcal{O}_{X_w}]_T = \sum_{v \preceq w} w_0(Q_{x,y}^\lambda)[\mathcal{O}_{X_v}]_T$$

where $Q_{x,y}^\lambda$ is as in (2.60). Hence substituting (2.60) in (2.61), we derive the ‘‘Chevalley formula’’ as in [18] and [17]. In particular, note that $w_0(Q_{w_0w, w_0v}^{w_0\lambda})$ is same as $C_{w,v}^\lambda$ of [18] where it is interpreted as $\sum e^{-\pi(1)}$ where the sum runs over all L-S paths π of shape λ ending in v and starting with an element smaller or equal to w . Here we briefly recall that an $L-S$ path π of shape λ on $X(\tau)$ is a pair of sequences $\pi = (\underline{\tau}, \underline{a})$ of Weyl group elements and rational numbers, where $\underline{\tau}$ is of the form $\underline{\tau} = (\tau_1, \dots, \tau_r)$ such that $\tau \geq \tau_1$ and $\tau_1 \geq \tau_2 \geq \dots \geq \tau_r$ in the Bruhat order on W . We call $\tau_1 = i(\pi)$ the ‘‘initial element of π and $\tau_r = e(\pi)$ the ‘‘end’’ element of π . (see §3 of [18]).

Remark 2.13.. We refer to [18] and [17] for more details on representation theoretic interpretation of the Chevalley formula using Standard Monomial Theory. We also refer to [11] for Chevalley formula in $K_T(G/B)$ given in terms of the combinatorics of the Littelmann path model, using techniques of affine nil Hecke-Algebra. Also see [22] for recent results on Chevalley formula in equivariant K -theory of flag varieties using the Bott-Samelson resolution.

2.4. Structure constants of Schubert basis in $K_T(G/P)$. In this section we determine a closed formula for the multiplicative structure constants of the Schubert basis $\{[\mathcal{O}_{X_P^w}]_T\}_{w \in W^I}$ of $K_T(G/P)$, again in terms of a_i, b_i .

Let μ^I be the Möbius function of the induced Bruhat ordering on W^I . Then (see Theorem 1.2 of [8]):

$$(2.62) \quad \mu^I(v, w) = \begin{cases} (-1)^{l(v)+l(w)} & \text{if } [v, w] \cap W^I = [v, w] \\ 0 & \text{otherwise} \end{cases}$$

where for $v \preceq w$, $[v, w] := \{u \in W : v \preceq u \preceq w\}$.

Lemma 2.14.. *For $x, y, z \in W^I$, let $D_{x,y}^z :=$*

$$(2.63) \quad \sum_{w \in W^I, w \preceq z} \mu^I(z, w) \sum_{1 \leq i, j \leq n} e^\rho \cdot a_i \cdot a_j \cdot L_w(L_{x^{-1}w_0}(b_i \cdot e^{-\rho}) \cdot L_{y^{-1}w_0}(b_j \cdot e^{-\rho}) \cdot e^\rho).$$

Then in $K_T(G/P)$ we have:

$$(2.64) \quad [\mathcal{O}_{X_P^x}]_T [\mathcal{O}_{X_P^y}]_T = \sum_{z \in W^I} D_{x,y}^z [\mathcal{O}_{X_P^z}]_T$$

for $x, y \in W^I$.

Proof: Let $\pi : G/B \rightarrow G/P$ be the canonical projection. Then we have:

$$(2.65) \quad \pi^*([\mathcal{O}_{X_P^w}]_T) = [\mathcal{O}_{X^w}]_T \text{ for } w \in W^I,$$

where $\pi^* : K_T(G/P) \rightarrow K_T(G/B)$ is the induced morphism.

For any $v \in W^I$ we also have (see Lemma 3.4 of [10]):

$$(2.66) \quad \pi^* \xi_P^v = \sum_{u \in W_I} \xi^{vu}.$$

Furthermore, for $w, v \in W^I$, we shall identify the elements ξ_P^v and $[\mathcal{O}_P^w]_T$ in $K_T(G/P)$ with their images in $K_T(G/B)$ under the injective morphism π^* .

Further, it follows from (1.1) that for $r \in R(T)^{W_I}$ and $\alpha \in I$ we have:

$$(2.67) \quad L_{s_\alpha}(r \cdot e^{-\rho}) = \frac{r \cdot e^{-\rho} - r \cdot e^{-\rho+\alpha}}{1 - e^\alpha} = r \cdot e^{-\rho}.$$

This implies that for any $(w', v) \in W_I \times W^I$ we have:

$$(2.68) \quad L_{vw'}(r \cdot e^{-\rho}) = L_v L_{w'}(r \cdot e^{-\rho}) = L_v(r \cdot e^{-\rho}).$$

Now, by (2.30), (2.66) and (2.67) it follows that for any $r \in R(T)^{W_I}$:

$$(2.69) \quad c_K^T(r) = \sum_{v \in W^I} e^\rho \cdot L_v(r \cdot e^{-\rho}) \cdot \xi_P^v.$$

Further, by Prop. 2.5 and (2.65) we have:

$$(2.70) \quad \Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 - e^\rho)) = [\mathcal{O}_{X_P^w}]_T.$$

By Lemma 2.6 we have:

$$L_{w^{-1}w_0}(e^{-\rho} \cdot b_i) \cdot e^\rho \in R(T)^{W_I} \text{ for } 1 \leq i \leq n$$

whenever $w \in W^I$.

Now, from (2.38), (2.66) and substituting $L_{w^{-1}w_0}(e^{-\rho}) \cdot b_i \cdot e^\rho$ for r in (2.68) we get:

$$(2.71) \quad [\mathcal{O}_{X_P^w}]_T = \Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho)) = \sum_{v \in W^I} \sum_{i=1}^n e^\rho \cdot a_i \cdot L_v(L_{w^{-1}w_0}(e^{-\rho} \cdot b_i)) \xi_P^v.$$

Let $\bar{w} \in W$ be such that $L_{\bar{w}} = L_v \cdot L_{w^{-1}w_0}$. Then by (2.34) we have:

$$(2.72) \quad \sum_{i=1}^n e^\rho \cdot a_i \cdot L_{\bar{w}}(e^{-\rho} \cdot b_i) = \delta_{\bar{w}, w_0}.$$

Further, by Lemma 2.4, (2.71) can be rewritten as:

$$(2.73) \quad [\mathcal{O}_{X_P^w}]_T = \Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho)) = \sum_{v \in W^I, w \preceq v} \xi_P^v.$$

Further, via Möbius inversion (2.73) is equivalent to:

$$(2.74) \quad \xi_P^v = \sum_{w \in W^I} \mu^I(v, w) [\mathcal{O}_{X_P^w}]_T$$

where $\mu^I(v, w)$ is as defined in (2.62).

Now, substituting (2.74) in (2.69) and using Lemma 2.1 we get:

$$(2.75) \quad \Psi(t \otimes r) = \sum_{w \in W^I} \mu^I(v, w) \cdot e^\rho \cdot t \cdot L_v(r \cdot e^{-\rho}) [\mathcal{O}_{X_P^w}]_T$$

for $t \otimes r \in R(T) \otimes R(T)^{W_I}$. Since, by Prop. 2.5 and Prop. 2.7 we have:

$$[\mathcal{O}_{X_P^w}] = \Psi(\mathbb{L}_{w^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho)),$$

(2.75) implies that:

$$(2.76) \quad \Psi(\mathbb{L}_{x^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho) \cdot \mathbb{L}_{y^{-1}w_0}(u_0) \cdot (1 \otimes e^\rho)) = [\mathcal{O}_{X_P^x}]_T [\mathcal{O}_{X_P^y}]_T$$

for $x, y \in W^I$.

Thus by (2.51), (2.75) and (2.76) we get (2.64), where $D_{x,y}^z$ is as in (2.63). Hence the lemma. \square

Remark 2.15.. Note that Lemma 2.14 is a generalization of Lemma 2.9 to partial flag varieties.

3. ANALOGOUS RESULTS IN ORDINARY K -THEORY

In this section we construct explicit lifts of structure sheaves of Schubert varieties in $K(G/B)$ in $R(T)$. Indeed, we do have the forgetful homomorphism $K_T(G/B) \rightarrow K(G/B)$ by which we could lift the classes $[\mathcal{O}_{X^w}]$ in $R(T) \otimes R(T)$. However, we require the lifts to be in $1 \otimes R(T)$ in order to apply these results in the next section to describe the Grothendieck ring of the wonderful compactification of an adjoint semi-simple group. It is not yet clear if the lifts in $R(T) \otimes R(T)$ via the forgetful homomorphism, can be chosen to be in this form. Since the nature of results and techniques used in the proofs are very similar to those in §2, we shall try here to be as brief in our arguments as possible.

Proposition 3.1.. *In $R(T)$ there exists an element u_0 such that*

$$(3.77) \quad c_K(L_{w^{-1}w_0}(u_0) \cdot e^\rho) = [\mathcal{O}_{X^w}]$$

where F is as in (1.16). Indeed we may take $u_0 = v_0 \cdot e^{-\rho}$ where $c_K(v_0) = [\mathcal{O}_{X^{w_0}}]$.

Proof: By (1.15) it follows that $a_{w_0} = [\mathcal{O}_{X^{w_0}}] = [\mathcal{O}_{pt}]$. Now, since c_K is surjective there exists an element v_0 such that $c_K(v_0) = a_{w_0}$. Hence by (1.13) it follows that

$$(3.78) \quad \epsilon L_w(v_0 \cdot e^{-\rho}) = \delta_{w,w_0}.$$

Claim: $u_0 = v_0 \cdot e^{-\rho}$ is the required element.

Proof of Claim: Let $\bar{w} \in W$ be as in (1.3). Then by (1.13) and (1.14) it follows that

$$(3.79) \quad c_K(L_{w^{-1}w_0}(u_0) \cdot e^\rho) = \sum_{w' \in W} \epsilon L_{w'} L_{w^{-1}w_0}(u_0) a_{w'} = \sum_{w' \in W} \epsilon L_{\bar{w}}(u_0) a_{w'}.$$

Now, we see that the claim follows by (3.78), (1.15) and Lemma 2.4. \square

Let u_0 be as in Prop. 3.1.

Proposition 3.2.. *The element $L_{w^{-1}w_0}(u_0) \cdot e^\rho \in R(T)^{W_I}$ if $w \in W^I$.*

Proof : This follows immediately by applying Lemma 2.6 for $r = v_0$. \square

We now state the results in ordinary K -ring analogous to Lemmas 2.9, 2.10, 2.14. Since the proofs follow the same lines as the proofs of the above mentioned lemmas, we avoid the repetition here.

Lemma 3.3.. For $x, y, z \in W$, let

$$(3.80) \quad c_{x,y}^z := \sum_{w \preceq z} (-1)^{l(z)-l(w)} \epsilon L_w(L_{x^{-1}w_0}(u_0) \cdot L_{y^{-1}w_0}(u_0) \cdot e^\rho).$$

Then in $K(G/B)$ we have:

$$(3.81) \quad [\mathcal{O}_{X^x}][\mathcal{O}_{X^y}] = \sum_{z \in W} c_{x,y}^z [\mathcal{O}_{X^z}]$$

for $x, y \in W$.

Lemma 3.4.. For $\lambda \in X^*(T)$ and $x, y \in W$ let

$$(3.82) \quad q_{x,y}^\lambda := \sum_{w \preceq y} (-1)^{l(y)-l(w)} \epsilon L_w(e^\lambda \cdot L_{x^{-1}w_0}(u_0)).$$

Then in $K(G/B)$ we have:

$$(3.83) \quad [\mathcal{L}(\lambda)][\mathcal{O}_{X^x}] = \sum_{y \in W} q_{x,y}^\lambda [\mathcal{O}_{X^y}].$$

Lemma 3.5.. For $x, y, z \in W^I$, let

$$(3.84) \quad d_{x,y}^z := \sum_{w \preceq z} \mu^I(z, w) \epsilon L_w(L_{x^{-1}w_0}(u_0) \cdot L_{y^{-1}w_0}(u_0) \cdot e^\rho).$$

Then in $K(G/P)$ we have:

$$(3.85) \quad [\mathcal{O}_{X_P^x}][\mathcal{O}_{X_P^y}] = \sum_{z \in W^I} d_{x,y}^z [\mathcal{O}_{X_P^z}]$$

for $x, y \in W^I$.

4. K -RING OF THE WONDERFUL COMPACTIFICATION

4.1. Some preliminaries. Let $\alpha_1, \dots, \alpha_r$ be an ordering of the set Δ of simple roots and $\omega_1, \dots, \omega_r$ denote the corresponding fundamental weights for the root system of (G, T) . Since G is simply connected, the fundamental weights form a basis for $X^*(T)$ and hence for every $\lambda \in \Lambda$, $e^\lambda \in R(T)$ is a Laurent monomial in the elements $e^{\omega_i} : 1 \leq i \leq r$.

In Theorem 2.2 of [19] Steinberg has defined a basis

$$(4.86) \quad \{f_v : v \in W^I\}$$

of $R(T)^{W_I}$ as a free $R(G)$ -module of rank $|W^I|$. We recall here this definition: For $v \in W^I$ let

$$p_v := \prod_{v^{-1}\alpha_i < 0} e^{\omega_i} \in R(T).$$

Then

$$f_v := \sum_{x \in W_I(v) \setminus W_I} x^{-1} v^{-1} p_v$$

where $W_I(v)$ denotes the stabilizer of $v^{-1}p_v$ in W_I .

Let $c_K : R(T)^{W_I} \rightarrow K(G/P_I)$ denotes the restriction of the characteristic homomorphism (see §8 of [14]). Let $I(G) := \{a - \epsilon(a) \mid a \in R(G)\}$ denote the augmentation ideal. Then it is known that c_K is surjective ring homomorphism and

$$(4.87) \quad \ker(c_K) = I(G) \cdot R(T)^{W_I}.$$

Let $r_v := L_{v^{-1}w_0}(u_0) \cdot e^\rho \in R(T)^{W_I}$ for $v \in W^I$ for every $I \subset \Delta$. Then by Lemma 3.2 we recall that $c_K(r_v) = [\mathcal{O}_{X_{P_I}^v}]$ for every $v \in W^I$.

Further, we recall that $R(T)^{W_I} = \mathbb{Z}[\Lambda]^{W_I}$ and $R(G) = R(T)^W = \mathbb{Z}[\Lambda]^W$.

Note that $\mathfrak{p} := I(G)$ is a prime ideal in $R(G)$ and let $R(G)_{\mathfrak{p}}$ denote the corresponding localization. We further observe that the augmentation extends to

$$(4.88) \quad R(G)_{\mathfrak{p}} \rightarrow \mathbb{Q}.$$

with kernel the maximal ideal $\mathfrak{p} \cdot R(G)_{\mathfrak{p}}$. Further, the characteristic homomorphism extends to

$$(4.89) \quad c_K : R(T)_{\mathfrak{p}}^{W_I} \twoheadrightarrow K(G/P_I)_{\mathbb{Q}}$$

with kernel $\mathfrak{p} \cdot R(T)_{\mathfrak{p}}^{W_I}$.

Lemma 4.1.. *The elements $\{r_v : v \in W^I\}$ form a basis of $R(T)_{\mathfrak{p}}^{W_I}$ as an $R(G)_{\mathfrak{p}}$ module.*

Proof: Now, $R(T)_{\mathfrak{p}}^{W_I}$ is a finitely generated $R(G)_{\mathfrak{p}}$ -module. Moreover, $\{c_K(r_v) = [\mathcal{O}_{X^v}]\}$, $v \in W^I$ form a basis of $K(G/P_I)_{\mathbb{Q}}$ as a $R(G)_{\mathfrak{p}}/\mathfrak{p} \cdot R(G)_{\mathfrak{p}} \simeq \mathbb{Q}$ -vector space. Now, by Nakayama lemma (see Prop. 2.8 of [1]) we see that $\{r_v : v \in W^I\}$ span $R(T)_{\mathfrak{p}}^{W_I}$ as an $R(G)_{\mathfrak{p}}$ -module. Furthermore, since $R(T)_{\mathfrak{p}}^{W_I}$ is free over $R(G)_{\mathfrak{p}}$ of rank $|W^I|$, it follows that $\{r_v : v \in W^I\}$ form a basis of $R(T)_{\mathfrak{p}}^{W_I}$ as an $R(G)_{\mathfrak{p}}$ -module. \square

We now fix some notations (also see p.378 of [21]).

Note that $J \subseteq I$ implies that $W^{\Delta \setminus J} \subseteq W^{\Delta \setminus I}$. Let

$$(4.90) \quad C^I := W^{\Delta \setminus I} \setminus \left(\bigcup_{J \subsetneq I} W^{\Delta \setminus J} \right)$$

and

$$(4.91) \quad R(T)_I := \bigoplus_{v \in C^I} R(T)_\mathfrak{p}^W \cdot r_v.$$

where $R(T)_\mathfrak{p}^W = R(G)_\mathfrak{p}$.

Lemma 4.2.. *We have the following direct sum decompositions as $R(T)^W$ modules:*

$$(4.92) \quad R(T)_\mathfrak{p}^{W^{\Delta \setminus I}} = \bigoplus_{J \subseteq I} R(T)_J$$

$$(4.93) \quad R(T)_\mathfrak{p}^{W^{\Delta \setminus I}} = \left(\sum_{J \subsetneq I} R(T)_\mathfrak{p}^{W^{\Delta \setminus J}} \right) \bigoplus R(T)_I$$

for $I \subseteq \Delta$.

Proof: By (4.90) we note that $W^{\Delta \setminus I} = \bigsqcup_{J \subseteq I} C^J$. Hence by Lemma 4.1 it follows that:

$$(4.94) \quad R(T)_\mathfrak{p}^{W^{\Delta \setminus I}} = \bigoplus_{J \subseteq I} \bigoplus_{v \in C^J} R(T)_\mathfrak{p}^W \cdot r_v.$$

Now using (4.91), the proof of the lemma follows exactly as that of Lemma 1.10 of [21]. \square

In $R(T)_\mathfrak{p}$ we have:

$$(4.95) \quad r_v \cdot r_{v'} = \sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} a_{v,v'}^w \cdot r_w$$

for certain elements $a_{v,v'}^w \in R(G)_\mathfrak{p} = R(T)_\mathfrak{p}^W$, $\forall v \in C^I$, $v' \in C^{I'}$ and $w \in C^J$ $J \subseteq (I \cup I')$.

Further, let λ_I denote the image of the element $\prod_{\alpha \in I} (1 - e^{-\alpha}) \in R(T)$ in $K(G/B)$ under c_K for every $I \subseteq \Delta$.

Further, let

$$K(G/B)_{\mathbb{Q}, I} := \bigoplus_{v \in C^I} \mathbb{Q} \cdot [\mathcal{O}_{X^v}].$$

Then we have:

$$K(G/B)_{\mathbb{Q}} = \bigoplus_{I \subseteq \Delta} K(G/B)_{\mathbb{Q}, I}.$$

4.2. Main Theorem. Let $X := \overline{G_{ad}}$ denote the wonderful compactification of the semisimple adjoint group $G_{ad} = G/Z(G)$, where $Z(G)$ denotes the center of G , constructed by De Concini and Procesi in [9].

Note that $K_{G \times G}(X)$ is an $\mathcal{R} := R(G) \otimes R(G)$ -module. Let

$$\mathcal{S} := R(G) \otimes R(G)_{\mathfrak{p}}.$$

Then we note that the forgetful homomorphism extends to

$$(4.96) \quad f : K_{G \times G}(X) \otimes_{\mathcal{R}} \mathcal{S} \rightarrow K(X)_{\mathbb{Q}}.$$

Theorem 4.3.. *The ring $K_{G \times G}(X) \otimes_{\mathcal{R}} \mathcal{S}$ has the following direct sum decomposition as an \mathcal{S} -module:*

$$(4.97) \quad K_{G \times G}(X) \otimes_{\mathcal{R}} \mathcal{S} = \bigoplus_{I \subseteq \Delta} \prod_{\alpha \in I} (1 - e^{\alpha(u)}) \cdot R(T) \otimes R(T)_I.$$

Further, the above direct sum is a free $R(T) \otimes R(G)_{\mathfrak{p}}$ -module of rank $|W|$ with basis

$$\left\{ \prod_{\alpha \in I} (1 - e^{\alpha(u)}) \otimes r_v : v \in C^I \text{ and } I \subseteq \Delta \right\},$$

where C^I is as defined in (4.90) and $\{r_v\}$ is as defined above. Moreover, we can identify the component $R(T) \otimes 1 \subseteq R(T) \otimes R(G)_{\mathfrak{p}}$ in the above direct sum with the subring of $K_{G \times G}(X)$ generated by $\text{Pic}^{G \times G}(X)$. (We refer to [20] for similar description of the equivariant cohomology ring of the wonderful compactifications.)

Proof: Recall from Lemma 3.2 of [21] that we have a chain of inclusions:

$$(4.98) \quad R(T) \otimes R(G) \subseteq K_{G \times G}(X) \subseteq R(T) \otimes R(T)$$

where $K_{G \times G}(X)$ consists of elements $f(u, v) \in R(T) \otimes R(T)$ that satisfy the relations

$$(4.99) \quad (1, s_{\alpha})f(u, v) \equiv f(u, v) \pmod{(1 - e^{\alpha(u)})} \text{ for every } \alpha \in \Delta.$$

Now, since $1 \otimes R(G)_{\mathfrak{p}}$ is a flat $1 \otimes R(G)$ -module, we see that \mathcal{S} is flat as an \mathcal{R} -module (see Ex 2.20 of [1]). This implies from (4.98) that we have a chain of inclusions:

$$(4.100) \quad (R(T) \otimes R(G)) \otimes_{\mathcal{R}} \mathcal{S} \subseteq K_{G \times G}(X) \otimes_{\mathcal{R}} \mathcal{S} \subseteq (R(T) \otimes R(T)) \otimes_{\mathcal{R}} \mathcal{S}.$$

Further, $K_{G \times G}(X) \otimes_{\mathcal{R}} \mathcal{S}$ consists of elements $f(u, v) \in (R(T) \otimes R(T)) \otimes_{\mathcal{R}} \mathcal{S}$ that satisfy the relations

$$(4.101) \quad (1, s_{\alpha})f(u, v) \equiv f(u, v) \pmod{(1 - e^{\alpha(u)})} \text{ for every } \alpha \in \Delta.$$

Here we simply use the fact that if $f(u, v) \in \mathcal{S}$, then $(1, s_\alpha)f(u, v) = f(u, v)$ for every $\alpha \in \Delta$. The theorem now follows by using Lemma 4.2 above, and replacing the Steinberg basis $\{f_v\}_{v \in W^I}$ by the canonical lifting of the Schubert basis $\{r_v\}_{v \in W^I}$, in the proof of Theorem 3.8 of [21]. \square

Theorem 4.4.. *The subring of $K(X)$ generated by classes of line bundles is isomorphic to $K(G/B)$. Moreover, $K(X)$ is a free module of rank $|W|$ over $K(G/B)$. More explicitly, let*

$$\gamma_v := 1 \otimes [\mathcal{O}_{X^v}] \in K(G/B) \otimes K(G/B)_{\mathbb{Q}, I}$$

for $v \in C^I$ for every $I \subseteq \Delta$. Then we have:

$$K(X)_{\mathbb{Q}} \simeq \bigoplus_{v \in W} K(G/B) \cdot \gamma_v.$$

Further, the above isomorphism is a ring isomorphism, where the multiplication of any two basis elements γ_v and $\gamma_{v'}$ is defined as follows:

$$\gamma_v \cdot \gamma_{v'} := \sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} (\lambda_{I \cap I'} \cdot \lambda_{(I \cup I') \setminus J} \cdot c_{v, v'}^w) \cdot \gamma_w.$$

where $c_{v, v'}^w \in \mathbb{Z}$ are as defined in (3.80).

Proof: Since $c_K(r_v) = [\mathcal{O}_{X^v}]$ for $v \in W^I$ and $I \subseteq \Delta$, we note that the image under c_K of the element $a_{v, v'}^w \in R(G)_{\mathfrak{p}}$ defined in (4.95) is nothing but the structure constant $c_{v, v'}^w \in \mathbb{Z}$ defined in (3.80). The proof now follows exactly as that of Theorem 3.12 on p.403 of [21]. \square

Remark 4.5.. Note that Theorem 4.4 is a restatement of Theorem 3.12 of [21], obtained by replacing the Steinberg basis $\{f_v\}_{v \in W^I}$ by the lift of the Schubert basis $\{r_v\}_{v \in W^I}$. In Theorem 3.12 of [21], the multiplicative structure of the ordinary K -ring of the wonderful group compactifications was described in terms of the structure constants of the image of the Steinberg basis $\{f_v\}_{v \in W^I}$ under c_K . These structure constants do not have any known relations to geometry or representation theory.

Whereas now we see that the multiplicative structure constants of the basis $\gamma_v = 1 \otimes [\mathcal{O}_{X^v}]$ of $K(X)_{\mathbb{Q}}$ as $K(G/B)$ -module are determined explicitly in terms of the multiplicative structure constants of the Schubert basis $c_K(r_v)$ described above in Prop. 3.3. These structure constants have been described in §2 and §3 above, and also are known to have nice geometric and representation theoretic interpretations (see for example [3] and [18]).

5. APPENDIX

Note that we can extend the definition of ϵ from $\mathbb{Q}[\Lambda] \rightarrow \mathbb{Q}$ and we can also extend the definition of Demazure operators to $\mathbb{Q}[\Lambda]$. Thus an element $u_0 \in \mathbb{Q}[\Lambda]$ which satisfies (3.78) will also satisfy (2.28), and will thereby give canonical lifting of $[\mathcal{O}_{X^w}]$ in the ring $\mathbb{Q}[\Lambda] \supset \mathbb{Z}[\Lambda]$. In the following proposition we shall explicitly determine one such canonical element u_0 in $\mathbb{Q}[\Lambda]$, in fact more precisely in $\mathbb{Z}[\frac{1}{|W|}][\Lambda]$.

Proposition 5.1.. *Let*

$$d = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w \in W} (\det w) w(e^\rho).$$

be the canonical anti-invariant, where $\det w = (-1)^{l(w)}$. We can then choose $u_0 = \frac{d}{|W|} \cdot \det(w_0)$ in the Prop. 3.1.

Proof: By (3.78) we need to show that $\epsilon L_w(\frac{d}{|W|} \cdot \det(w_0)) = \delta_{w, w_0}$. Note that

$$(5.102) \quad d = \det(w_0) e^{-\rho} \prod_{\alpha \in \Phi^+} (1 - e^\alpha).$$

We shall first prove that $\epsilon L_{w_0}(u_0) = 1$ where $u_0 := \frac{d}{|W|} \cdot \det(w_0)$.

Let

$$(5.103) \quad J := \sum_{w \in W} \det(w) \cdot w$$

as in pp.183-186 of [2] or p.87 of [7]. Then $J(e^\rho) = d$. Moreover, since $e^{-\rho} = w_0(e^\rho)$, we have:

$$\begin{aligned} J(e^{-\rho}) &= \sum_{w \in W} \det(w) \cdot w w_0(e^\rho) \\ &= \sum_{w \in W} \det(w w_0) \cdot w w_0(e^\rho) \\ (5.104) \quad &= \det(w_0) \cdot \sum_{w' \in W} \det(w') \cdot w'(e^\rho) \\ &= \det(w_0) \cdot J(e^\rho) \\ &= \det(w_0) \cdot d. \end{aligned}$$

Recall from (16) on p.87 of [7] that:

$$(5.105) \quad L_{w_0} = e^{-\rho} [J(e^{-\rho})]^{-1} \cdot J$$

Now, substituting (5.104) in (5.105) we get:

$$(5.106) \quad L_{w_0} = e^{-\rho} \cdot \det(w_0)^{-1} \cdot d^{-1} \cdot J.$$

This further implies that:

$$(5.107) \quad \begin{aligned} L_{w_0}(u_0) &= e^{-\rho} \cdot \det(w_0)^{-1} \cdot d^{-1} \cdot J(u_0) \\ &= e^{-\rho} \cdot \det(w_0)^{-1} \cdot d^{-1} \cdot \det(w_0) \cdot \frac{d}{|W|} \cdot |W| \\ &= e^{-\rho}. \end{aligned}$$

Hence we have: $\epsilon L_{w_0}(u_0) = \epsilon(e^{-\rho}) = 1$.

Now we shall show that $\epsilon L_w(u_0) = 0$ for every $w \neq w_0$. We first fix some notations.

Let I be the ideal in $\mathbb{Q}[\Lambda]$ generated by $\{1 - e^\alpha : \alpha \in \Phi^+\}$. Then I^m is generated by $\{\prod_{1 \leq i \leq m} (1 - e^{\gamma_i}) : \gamma_i \in \Phi^+ \forall i\}$ for $m \geq 1$. By (5.102) it follows that $d \in I^{l(w_0)}$.

We claim that for $f \in I^m$ and $\gamma \in \Delta$, $L_{s_\gamma}(f) \in I^{m-1}$. To prove the claim, it is enough to take f to be a generator of I^m so that we can express it as $(1 - e^\alpha) \cdot g$ where $g \in I^{m-1}$. We can now apply induction on m (note that since $I^0 = \mathbb{Q}[\Lambda]$, the claim clearly holds for $m = 1$). It is easy to see that (see p. 344 of [14]):

$$(5.108) \quad L_{s_\gamma}((1 - e^\alpha) \cdot g) = g \cdot L_{s_\gamma}(1 - e^\alpha) + (1 - e^\alpha) \cdot L_{s_\gamma}(g) - (1 - e^\gamma) \cdot L_{s_\gamma}(1 - e^\alpha) \cdot L_{s_\gamma}(g).$$

Now, by induction assumption we have: $L_{s_\gamma}(g) \in I^{m-2}$. Hence by (5.108) it follows that $L_{s_\gamma}((1 - e^\alpha) \cdot g) \in I^{m-1}$. This proves the claim.

Moreover, if $w = s_{i_1} \cdot s_{i_2} \cdots s_{i_k}$ is a reduced decomposition of w then by (1.2), $L_w = L_{s_{i_1}} \cdots L_{s_{i_k}}$. Thus recursively applying the above arguments it follows that for $f \in I^m$, where $l(w) = k \geq m$, $L_w(f) \in I^{m-k}$.

Since $u_0 = \frac{d}{|W|} \cdot \det(w_0) \in I^{l(w_0)}$, we have: $L_w(u_0) \in I^{l(w_0)-l(w)}$ for every $w \in W$. If $w \neq w_0$ then $l(w) < l(w_0)$, so that $L_w(u_0) \in I$. Clearly $I \subseteq \ker(\epsilon)$. Thus it follows that $\epsilon L_w(u_0) = 0$ if $w \neq w_0$ which proves the proposition. \square

Remark 5.2.. Let $\{e^{p_w}\}_{w \in W}$ be the basis defined by Steinberg of $R(T)$ as an $R(T)^W$ -module where:

$$p_w = w \left(\sum_{\alpha \in \Delta, w(\alpha) < 0} \omega_\alpha \right)$$

for $w \in W$ (see [19]). Then by lemme 4 and prop. 3 of [14] we see that the matrix $M = (L_{w'}(e^{p_w}))_{w, w' \in W}$ with entries in $R(T)$ is invertible. Thus there exists a unique vector $(a_w)_{w \in W}$ such that

$$\sum_{w \in W} a_w \cdot L_{w'}(e^{p_w}) = e^{-\rho} \cdot \delta_{w', w_0}$$

for every $w' \in W$. Now, defining $b_w := e^{\rho + p_w}$, we see that the element

$$v_0 = \sum_{w \in W} a_w \otimes b_w$$

in $R(T) \otimes R(T)$ satisfies (2.34). Thus we have a canonical choice of an element

$$u_0 = v_0 \cdot (1 \otimes e^{-\rho}) = \sum_{w \in W} a_w \otimes e^{p_w}$$

in $R(T) \otimes R(T)$ which satisfies Prop. 2.5. The author is currently working on finding a more explicit candidate for u_0 in $R(T) \otimes R(T)$ satisfying Prop.2.5.

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